APPENDIX D

GREAT CIRCLE CALCULATIONS

A simple and direct method of performing the great-circle calculations required in siting line-of-sight and scatter communication stations is presented here. An understanding of how the method is derived is not needed.

The calculation of the great-circle path length and azimuths between the transmitter and the receiver sites can be easily made if the latitudes and longitudes of the sites are known. Usually these coordinates can be obtained with sufficient accuracy from reliable maps of the areas involved. It is worthwhile for two persons to make the computations independently, comparing their results after each step. If only one person is making the computations, he should check each step thoroughly as it is completed. The computations require the addition, subtraction, multiplication, and divisions of positive and negative numbers and the use of tables of functions.

An accuracy of two minutes of arc is usually adequate for the great-circle calculations needed in siting line-of-sight scatter communication stations. This accuracy can be obtained by using five-place tables of logarithms and trigonometric functions. The tables should be graduated for every minute or every one-hundredth of a degree of arc. If five-place tables are used, it is recommended that six decimal places be carried in performing the arithmetical operations of the great-circle calculations and that interpolation be used to obtain all functions to six places and all angles to the nearest tenth of a minute or one-thousandth of a degree of arc. The computed azimuths should then be rounded to the nearest minute or one-hundredth of a degree of arc and the path length should be rounded to one decimal place, although the last digit will not always be significant.

The procedure presented here requires the uniform system of nomenclature shown in figure D-1. The location having the more westerly longitude is designated point A and the location having the more easterly longitude as point B. The North Pole is always used as the third point of the terrestrial triangle, regardless of the latitude of point A or B, and is designated as P.

The equations upon which the great-circle computations are based are the law of cosines. In terms of the terrestrial triangle PAB shown in figure D-1, these are

$$\frac{\sin A}{\sin a} = \frac{\sin P}{\sin p} = \frac{\sin B}{\sin B} \qquad \text{(D-1)}$$

and

$$\cos p = \cos a \cos b + \sin a \sin b \cos P \qquad \text{(D-2)}$$

where equation (D-1) is the law of sines and equation (D-2) is the law of cosines. Since $a = (90^\circ - \text{Lat B})$ and $b = (90^\circ - \text{Lat A})$, equations (D-1) and (D-2) may be rewritten as

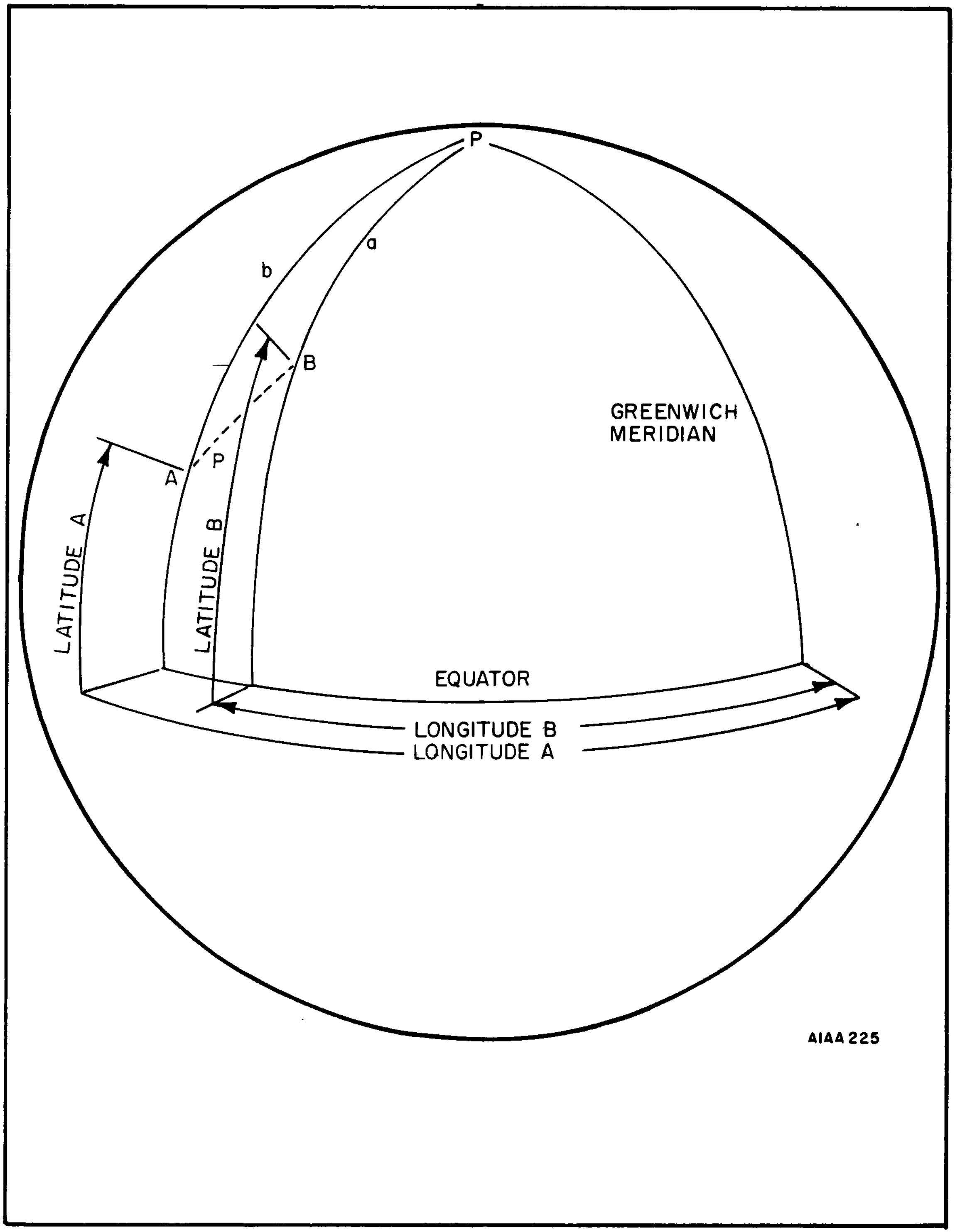

Figure D-1. Geometry for Great-Circle Calculations

$$\cos p = \sin(\text{Lat B})\sin(\text{Lat A}) + \cos(\text{Lat B})\cos(\text{Lat A})\cos P \qquad \text{(D-3)}$$

and

$$\frac{\sin A}{\cos(\text{Lat B})} = \frac{\sin P}{\sin p} = \frac{\sin B}{\cos(\text{Lat A})}$$

from which the following equations for sin A and sin B are obtained:

$$\sin A = \frac{\sin P}{\sin p}\cos(\text{Lat B}) \qquad \text{(D-4)}$$

and

$$\sin B = \frac{\sin P}{\sin p}\cos(\text{Lat B}) \qquad \text{(D-5)}$$

Application of the law of cosines to angles A and B results in the equations

$$\cos A = \frac{\sin(\text{Lat B}) - \cos p \sin(\text{Lat A})}{\sin p \cos(\text{Lat A})} \qquad \text{(D-6)}$$

and

$$\cos B = \frac{\sin(\text{Lat A}) - \cos p \sin(\text{Lat B})}{\sin p \cos(\text{Lat B})} \qquad \text{(D-7)}$$

Angle P is equal to the algebraic difference of the longitudes of points A and B; that is, angle P = Long A - Long B and can have any value between 0^o and 360^o. Longitudes west of Greenwich are considered to be positive and those east of Greenwich are nagative. Likewise, latitudes north of the equator are positive and those south of the equator are negative. An arc above two capital letters is used to indicate the shorter great-circle are between two points on the earth, and the direction of the arc is indicated by the order in which the capital letters are written. For example, AB represents, and is read as, "the shorter great-circle arc from point A to point B."

For the purpose of this appendix, the azimuth at point A of AB is defined to be the angle at A between AP and AB, measured eastward from north. The azimuth at point B of BA is defined in a similar manner. These azimuths may have any value between 0^o and 360^o. For example, in figure D-1, the azimuth at point A of AB is the interior angle A, while the azimuth at point of B of BA is 360^o minus the interior angle B. In general, these azimuths do not differ by 180^o.

Angles A and B are special angles introduced to simplify the computation of azimuth. They are positive angles between 0^o and 90^o and are, by definition, equal to the values of angles A and B respectively, when these latter angles are obtained directly from the tables without regard to quadrant. Use of the trigonometric tables is thus simplified and large angles need not be dealt with until the last steps of the computation. For example, if sin A = -0.5, then A = 30^o, not -30^o. Similarly, if cos B = 0.5, then B = 60^o, not 120^o. The computation of both the sines and cosines of angles A and B from equations (D-4), (D-5), (D-6), and (D-7) allows the formulation of rules for the summarized in the tables below and on the computation forms.

Two examples are given, using forms especially designed to facilitate the computations. One illustrates the use of five-place tables of logarithms, the other a calculating machine (figure D-2 and figure D-3). The latter is also applicable if it is necessary to make the calculations by longhand (see figure D-4). If accurately followed, the indicated procedures will automatically eliminate any ambiguity in the quadrant of angles computed.

A typical computer program which may be used in great-circle calculations is shown in figure D-5. This program has been developed by the INFONET Division of Computer Sciences Corporation and it is reprinted here with their permission.

Given:

	Site	Latitude*	Longitude*
A.	Singapore	+01°18'N	-103°51'E
B.	Bali	-03°06'S	-115°05'E

I. Solve for P.

P = Long A - Long B = 11°14'

log sin P = (+)9.28960 - 10 . . . . (1)

II. Solve for p.

		Log Sine	Log Cosine	
Lat A =	+01°18'	(+)8.35578 - 10	(2) (+)9.99989 - 10	(4)
Lat B =	-08°06'	(-)9.14891 - 10	(3) (+)9.99565 - 10	(5)
P =	11°14'		(+)9.99160 - 10	(6)

(2) + (3) = (-)17.50469 - 20 . . . . (7)

(4) + (5) + (6) = 29.98714 - 30 (8)

antilog of (7) = -0.00320 . . . . (9)

antilog of (8) = 0.97083 . . . . (10)

cos p = (9) + (10) = 0.96763 . . . . (11)

p = 14°37.1' . . . . (12)

log sin p = 9.40205 - 10 . . . . (13)

log cos p = 9.98571 - 10 . . . . (14)

III. Solve for $\tilde{A}$ and $\tilde{B}$ from log sines of A and B.†

log sin A = (1) + (5) - (13) = 9.88320 - 10 . . . . (15)

$\tilde{A}$ = 49°50.1' . . . . (16)

log sin B = (1) + (4) - (13) = 9.88744 - 10 . . . . (17)

$\tilde{B}$ = 50°30.3' . . . . (18)

IV. Solve for $\tilde{A}$ and $\tilde{B}$ from cosines of A and B.†

cos A = antilog [(3) - (13) - (4)]
- antilog [(14) + (2) - (13) - (4)] = -0.64544 (19)

$\tilde{A}$ = 49°48.1' . . . . (20)

cos B = antilog [(2) - (13) - (5)]
- antilog [(14) + (3) - (13) - (5)] = 0.63646 (21)

$\tilde{B}$ = 50°28.5' . . . . (22)

V. Compute the azimuths using the following tables.

Log Sin A	Cos A	Azimuth at A of $\widehat{AB}$
+	+	$\tilde{A}$
+	-	180° - $\tilde{A}$
-	+	180° + $\tilde{A}$
-	+	360° - $\tilde{A}$

Log Sin B	Cos B	Azimuth at B of $\widehat{BA}$
+	+	360° - $\tilde{B}$
+	-	180° + $\tilde{B}$
-	-	180° - $\tilde{B}$
-	+	$\tilde{B}$

If $0° \le \tilde{A} \le 10°$, use (16) for $\tilde{A}$.
If $10° < \tilde{A} < 80°$, use either (16) or (20) for $\tilde{A}$.
If $80° \le \tilde{A} \le 90°$, use (20) for $\tilde{A}$.

If $0° \le \tilde{B} \le 10°$, use (18) for $\tilde{B}$.
If $10° < \tilde{B} < 80°$, use either (18) or (22) for $\tilde{B}$.
If $80° \le \tilde{B} \le 90°$, use (22) for $\tilde{B}$.

Azimuth at A of $\widehat{AB}$ = 130°10'

Azimuth at B of $\widehat{BA}$ = 309°30'

VI. Compute path length.

p = (12) = 14°37.1' = 877.1 minutes of arc . . . . (23)

log (23) = 2.94305 . . . . (24)

Statute miles = antilog of [(24) + 0.061302] = 1010.1

Kilometers = antilog of [(24) + 0.267932] = 1025.5

Nautical miles = (23) = 877.1

*Latitudes north of the equator are considered to be positive and those south to be negative. Similarly, longitudes west of Greenwich are considered to be positive and those east to be negative.

†Angles $\tilde{A}$ and $\tilde{B}$ are positive angles between 0° and 90° and are by definition, equal to the values of angles A and B respectively, when these latter angles are obtained directly from the tables without regard to quadrant.

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Figure D-2. Great-Circle Calculations, Using a Computer Machine

Given:

	Site	Latitude*	Longitude*
A.	Singapore	+01°18'N	-103°51'E
B.	Bali	-03°06'S	-115°05'E

I. Solve for P.

P = Long A - Long B = 11°14'

sin P = 0.19481 .... (1)

II. Solve for p.

Lat A =	+01°18'	0.022690	(2) 0.999740	(4)
Lat B =	-08°06'	-0.140900	(3) 0.990020	(5)
P =	11°14'		0.980840	

cos p = [(2) x (3)] + [(4) x (5) x (6)] = 0.967602 (7)

p = 14°37.5' .... (8)

sin p = 0.252490 .... (9)

III. Solve for $\tilde{A}$ and $\tilde{B}$ from sines of A and B.†

$\frac{\sin P}{\sin p} = \frac{(1)}{(9)} = 0.771555$ .... (10)

sin A = (10) x (5) = 0.763855 .... (11)

$\tilde{A}$ = 49°48.3' .... (12)

sin B = (10) x (4) = 0.771354 .... (13)

$\tilde{B}$ = 50°28.5' .... (14)

IV. Solve for $\tilde{A}$ and $\tilde{B}$ from cosines of A and B.†

$\cos A = \frac{(3) - [(7) \times (2)]}{(9) \times (4)} = -0.645104$ (15)

A = 49°49.3' .... (16)

$\cos B = \frac{(2) - [(7) \times (3)]}{(9) \times (5)} = 0.636176$ (17)

B = 50°89.6' .... (18)

V. Compute the azimuths using the following tables.

Sin A (11)	Cos A (15)	Azimuth at A of $\widehat{AB}$
+	+	$\tilde{A}$
+	-	180° - $\tilde{A}$
-	-	180° + $\tilde{A}$
-	+	360° - $\tilde{A}$

Sin B (13)	Cos B (17)	Azimuth at B of $\widehat{BA}$
+	+	360° - $\tilde{B}$
+	-	180° + $\tilde{B}$
-	-	180° - $\tilde{B}$
-	+	$\tilde{B}$

If $0° \leq \tilde{A} \leq 10°$, use (12) for $\tilde{A}$.
If $10° < \tilde{A} < 80°$, use either (12) or (16) for $\tilde{A}$.
If $80° \leq \tilde{A} \leq 90°$, use (16) for $\tilde{A}$.

If $0° \leq \tilde{B} \leq 10°$, use (14) for $\tilde{B}$.
If $10° < \tilde{B} < 80°$, use either (14) or (18) for $\tilde{B}$.
If $80° \leq \tilde{B} \leq 90°$, use (18) for $\tilde{B}$.

Azimuth at A of $\widehat{AB}$ = 130°11'

Azimuth at B of $\widehat{BA}$ = 309°30'

VI. Compute path length.

p = (8) = 14°57.5' = 877.5 minutes of arc .... (19)

Statute Miles = (19) x 1.1516 = 1010.5

Kilometers = (19) x 1.85325 = 1626.2

Nautical Miles = (19) = 877.5

*Latitudes north of the equator are considered to be positive and those south to be negative. Similarly, longitudes west of Greenwich are considered to be positive and those east to be negative.

† Angles $\tilde{A}$ and $\tilde{B}$ are positive angles between 0° and 90° and are by definition, equal to the values of angles A and B respectively, when these latter angles are obtained directly from the tables without regard to quadrant.

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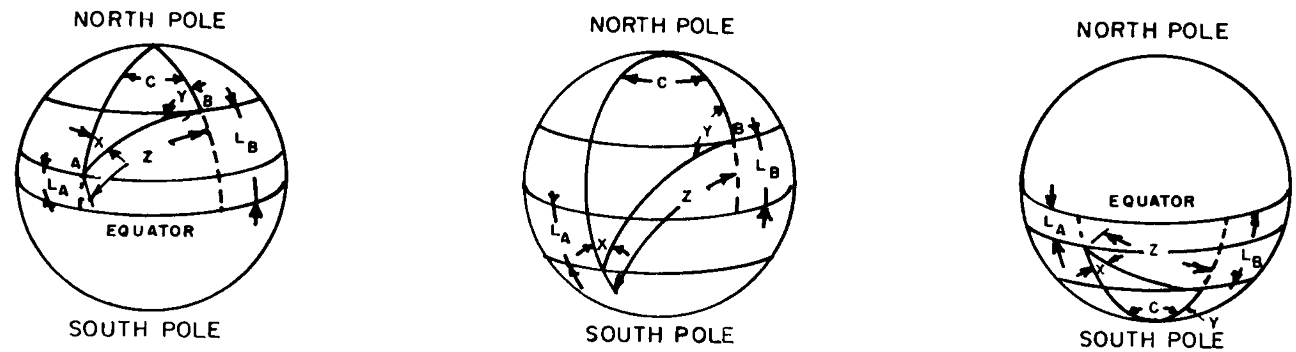
Figure D-3. Great-Circle Calculations, Using Logarithms



THREE GLOBES REPRESENTING POINTS A AND B BOTH IN THE NORTHERN HEMISPHERE, IN OPPOSITE HEMISPHERES, AND BOTH IN THE SOUTHERN HEMISPHERE. IN ALL CASES, L_A = LATITUDE OF B, C = DIFFERENCE OF LONGITUDE.

$$\text{TAN } \tfrac{1}{2}(Y-X) = \text{COT } \tfrac{1}{2}C \, \frac{\text{SIN } \tfrac{1}{2}(L_B - L_A)}{\text{COS } \tfrac{1}{2}(L_B + L_A)} \quad \text{AND} \quad \text{TAN } \tfrac{1}{2}(Y+X) = \text{COT } \tfrac{1}{2}C \, \frac{\text{COS } \tfrac{1}{2}(L_B - L_A)}{\text{SIN } \tfrac{1}{2}(L_B + L_A)}$$

$$\tfrac{1}{2}(Y+X) + \tfrac{1}{2}(Y-X) = Y$$

$$\tfrac{1}{2}(Y+X) - \tfrac{1}{2}(Y-X) = X$$

$$\text{TAN } \tfrac{1}{2}Z = \text{TAN } \tfrac{1}{2}(L_B - L_A) \; [\text{SIN } \tfrac{1}{2}(Y+X)] \; [\text{SIN } \tfrac{1}{2}(Y-X)]$$

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Figure D-4. Great-Circle Calculations (Sheet 1 of 3)

Site ______ ______° ______' ______" N Lat ______° ______' ______" W Long

Site ______ ______° ______' ______" N Lat ______° ______' ______" W Long

C = ______° ______' ______"

$L_B + L_A$ = ______° ______' ______"

$L_B - L_A$ = ______° ______' ______"

$\frac{L_B + L_A}{2}$ = ______° ______' ______"

$\frac{L_B - L_A}{2}$ = ______° ______' ______"

$\frac{C}{2}$ = ______° ______' ______"

$\log \cot (\frac{C}{2})$ ______° ______' ______" = ____________

$+ \log \cos \frac{L_B - L_A}{2}$ ______° ______' ______" = ____________

$- \log \sin \frac{L_B + L_A}{2}$ ______° ______' ______" = ____________

$\log \tan \frac{X + Y}{2}$ = ____________

$\frac{X + Y}{2}$ = ______° ______' ______"

$\log \cot (\frac{C}{2})$ ______° ______' ______" = ____________

$+ \log \sin \frac{L_B - L_A}{2}$ ______° ______' ______" = ____________

= ____________

$- \log \cos \frac{L_B + L_A}{2}$ ______° ______' ______" = ____________

$\log \tan \frac{X - Y}{2}$ = ____________

$\frac{X - Y}{2}$ = ____________

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Figure D-4. Great-Circle Calculations (Sheet 2 of 3)

Azimuth from Site ______________

$\frac{X + Y}{2}$ = ______° ______' ______"

$\frac{+ X - Y}{2}$ = ______° ______' ______"

____

______° ______' ______"

Azimuth from Site ______________

$\frac{X + Y}{2}$ = ______° ______' ______"

$\frac{- X - Y}{2}$ = ______° ______' ______"

____

______° ______' ______"

$\log \tan \frac{L_B - L_A}{2}$ ______° ______' ______" = ______________________

$+ \log \sin \frac{X + Y}{2}$ ______° ______' ______" = ______________________

$- \log \sin \frac{X - Y}{2}$ ______° ______' ______" = ______________________

$\log \tan \frac{Z}{2}$ = ______________________

$\frac{Z}{2}$ = ______° ______' ______"

Z = ______° ______' ______"

d = ______° x 69.093 = (Statute Miles) ______

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Figure D-4. Great-Circle Calculations (Sheet 3 of 3)

***SPHERE
SPHERICAL TRIANGLES
Geometry 02.03
Program No. 01-0640

ABSTRACT

***SPHERE solves spherical triangles having an apex at the North Pole and the other two corners defined by their respective latitudes and longitudes. Program output includes great circle distances, true bearings, and the hour angle at the North Pole.

DESCRIPTION

***SPHERE solves spherical triangles having an apex at the North Pole. Input consists of:

- Local latitude and longitude
- Remote latitude and longitude
- Observed altitude, if any

Output consists of:

- Local hour angle at the North Pole
- Zenith (great circle) distances
- True bearings (great circle courses)
- Altitude (remote celestial position above the local horizon)
- Line of position

INSTRUCTIONS

After the program description has printed, enter input data in the following format:

10 DATA LTD, LTM, LGD, LGM, RLTD, RLTM, RLGD, RLGM, ALD, ALM

E00004-00.016-00

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Figure D-5. Great-Circle Distance, Computer Program (Sheet 1 of 5)

***SPHERE

SPHERICAL TRIANGLES

where

LTD, LTM	= local latitude (degrees, minutes)
LGD, LGM	= local longitude (degrees, minutes)
RLTD, RLTM	= remote latitude (degrees, minutes)
RLGD, RLGM	= remote longitude (degrees, minutes)
ALD, ALM	= observed altitude (if any) (degrees, minutes)

Each pair of numbers specifies the degrees and the minutes of each associated location.

For South latitudes and East longitudes, enter the degree values as negative numbers. If there is no observed altitude, set ALD and ALM equal to zero.

The first DATA statement used must be numbered 10. DATA for as many cases as desired can be entered successively in succeeding DATA statements. DATA statements can be numbered 10-99.

After all DATA statements have been entered, type RUN (followed by a carriage return) and program execution will continue. To re-execute the program, enter the desired new DATA statements and type RUN again.

SAMPLE RUN

Solve the spherical triangle problem using the following data:

Local Latitude: 40 degrees 50 minutes North Latitude

Local Longitude: 73 degrees 30 minutes West Longitude

Remote Latitude: 23 degrees 26 minutes North Latitude

Remote Longitude: 133 degrees 30 minutes West Longitude

Observed Altitude: 37 degrees 20 minutes

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Figure D-5. Great-Circle Distance, Computer Program (Sheet 2 of 5)

***SPHERE
SPHERICAL TRIANGLES

```
RUN ***SPHERE

***SPHERE      11:39      05/12/70

        SØLUTIØN ØF SPHERICAL TRIANGLES
  #01-0640; VERSIØN 2

DETAILS (YES,NØ) ?YES


     ***SPHERE SØLVES SPHERICAL TRIANGLES HAVING THE
APEX AT THE NØRTH PØLE AND THE ØTHER TWØ CØRNERS DEFINED
BY THEIR RESPECTIVE LATITUDES AND LØNGITUDES.
MULTIPLE CASES MAY BE ENTERED SUCESSIVELY IN DATA
STATEMENTS 10-999 IN THE FØLLØWING FØRMAT:

   10 DATA  LTD,LTM, LGD,LGM, RLTD,RLTM, RLGD,RLGM, ALD,ALM

WHERE EACH PAIR ØF NUMBERS SPECIFIES A LØCATIØN IN THE FØRM
'DEGREES,MINUTES' AS FØLLØWS:

   LTD,LTM     = LØCAL LATITUDE
   LGD,LGM     = LØCAL LØNGITUDE
   RLTD,RLTM   = REMØTE LATITUDE
   RLGD,RLGM   = REMØTE LØNGITUDE
   ALD,ALM     = ØBSERVED ALTITUDE (IF ANY)

SØUTH LATITUDES AND EAST LØNGITITUDES ARE SPECIFIED
WITH NEGATIVE DEGREES AND PØSITIVE MINUTE VALUES.
IF THERE IS NØ ØBSERVERED ALTITUDE, SET ALD AND ALM
EQUAL TØ ZERØ.

END ØF ***SPHERE
NØW AT *END*

11:41    RAN 0 MINS  0.32 SECS



READY
10 DATA 40,50,73,30,23,26,133,30,37,20
RUN

***SPHERE      11:42      05/12/70

        SØLUTIØN ØF SPHERICAL TRIANGLES
  #01-0640; VERSIØN 2
```

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Figure D-5. Great-Circle Distance, Computer Program (Sheet 3 of 5)

```
***SPHERE
SPHERICAL TRIANGLES


   CASE NUMBER 1

   LØCAL PØSITIØN:

    40 DEG     50 MIN   NØRTH LATITUDE
    73 DEG     30 MIN   WEST LØNGITUDE

   REMØTE PØSITIØN:

    23 DEG     26 MIN   NØRTH LATITUDE
    133 DEG     30 MIN   WEST LØNGITUDE

   LØCAL HØUR ANGLE (AT NØRTH PØLE):

    60 DEG
    60 DEG     0 MIN
    4 HRS    0 MIN     0 SEC

   ZENITH (GREAT CIRCLE) DISTANCES:

    52.6 DEG
    52 DEG     37 MIN
    3157 NAUTICAL MILES
    3635.5  STATUTE MILES

   TRUE BEARINGS (GREAT CIRCLE CØURSES):

    REMØTE PØSITIØN FRØM LØCAL PØSITIØN:
    270.1 DEG
    270 DEG     4 MIN

    LØCAL PØSITIØN FRØM REMØTE PØSITIØN:
    55.6 DEG
    55 DEG     33 MIN

   ALTITUDE (REMØTE CELESTIAL PØSITIØN
      ABØVE LØCAL PØSITIØN HØRIZØN):

    37.4 DEG
    37 DEG     23 MIN
```

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Figure D-5. Great-Circle Distance, Computer Program (Sheet 4 of 5)

***SPHERE
SPHERICAL TRIANGLES

```
ØBSERVED ALTITUDE:

 37 DEG     20 MIN
 37.33 DEG


LINE ØF PØSITIØN:

 3 MILES AWAY  ØN LINE BEARING    90.1 DEGREES TRUE

END ØF ***SPHERE
NØW AT *END*

11:43    RAN 0 MINS  0.12 SECS
```

E00004-00.016-00

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Figure D-5. Great-Circle Distance, Computer Program (Sheet 5 of 5)